Problem A.15

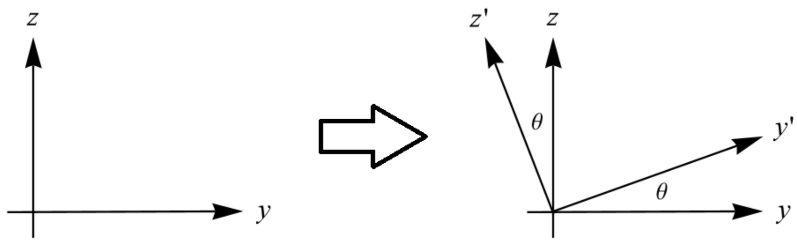
In the usual basis $\left(\hat{\imath}, \hat{\jmath}, \hat{k}\right)$, construct the matrix T_x representing a rotation through angle θ about the x axis, and the matrix T_y representing a rotation through angle θ about the y axis. Suppose now we change bases, to $\hat{\imath}' = \hat{\jmath}$, $\hat{\jmath}' = -\hat{\imath}$, $\hat{k}' = \hat{k}$. Construct the matrix S that effects this change of basis, and check that $\mathsf{S}\mathsf{T}_x\mathsf{S}^{-1}$ and $\mathsf{S}\mathsf{T}_y\mathsf{S}^{-1}$ are what you would expect.

Solution

The goal is to find T_x and T_y, the matrices representing the prescribed linear transformations with respect to the standard basis. (See Equation A.42 on page 470.)

$$\mathsf{a}' = \mathsf{T}\mathsf{a} \tag{A.42}$$

To find T_x, draw an xyz-coordinate system, looking at the origin from the positive x-axis.



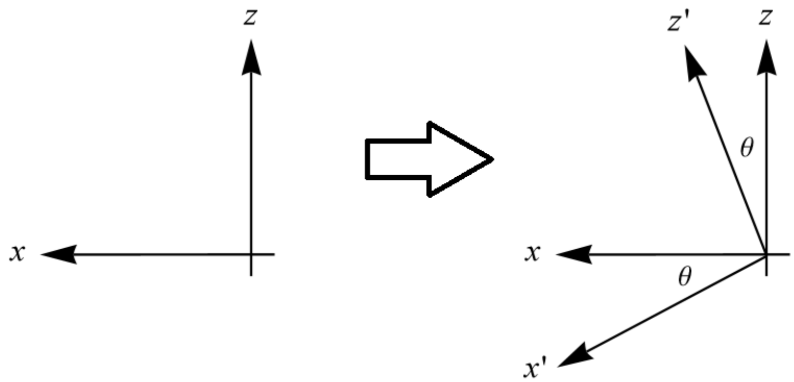
A counterclockwise rotation about the x-axis of angle θ results in the axes as shown on the right. The new unit vectors are expressed in terms of the old ones as follows.

$$\begin{cases} \hat{\imath}' = \hat{\imath} \\ \hat{\jmath}' = \cos\theta\,\hat{\jmath} + \sin\theta\,\hat{k} \\ \hat{k}' = -\sin\theta\,\hat{\jmath} + \cos\theta\,\hat{k} \end{cases} \quad \Rightarrow \quad \begin{bmatrix} \hat{\imath}' \\ \hat{\jmath}' \\ \hat{k}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix}$$

Therefore,

$$\mathsf{T}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}.$$

To find T_y, draw an xyz-coordinate system, looking at the origin from the positive y-axis.



A counterclockwise rotation about the y-axis of angle θ results in the axes as shown on the right. The new unit vectors are expressed in terms of the old ones as follows.

$$\begin{cases} \hat{i}' = \cos\theta\,\hat{i} - \sin\theta\,\hat{k} \\ \hat{j}' = \hat{j} \\ \hat{k}' = \sin\theta\,\hat{i} + \cos\theta\,\hat{k} \end{cases} \quad \Rightarrow \quad \begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

Therefore,

$$\mathsf{T}_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

Equation A.63 on page 474 gives the formula for a change of basis.

$$\mathsf{a}^f = \mathsf{S}\mathsf{a}^e \tag{A.63}$$

The prescribed change of basis is

$$\begin{cases} \hat{\imath}' = \hat{\jmath} \\ \hat{\jmath}' = -\hat{\imath} \\ \hat{k}' = \hat{k} \end{cases} \quad \Rightarrow \quad \begin{bmatrix} \hat{\imath}' \\ \hat{\jmath}' \\ \hat{k}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix}.$$

Therefore,

$$\mathsf{S} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which means

$$\mathsf{S}^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now calculate $\mathsf{S}\mathsf{T}_x\mathsf{S}^{-1}$ and $\mathsf{S}\mathsf{T}_y\mathsf{S}^{-1}$.

$$\mathsf{S}\mathsf{T}_x\mathsf{S}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \mathsf{T}_y(-\theta)$$

$$\mathsf{S}\mathsf{T}_y\mathsf{S}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = \mathsf{T}_x(\theta)$$

The first equation says that making a counterclockwise rotation of θ about the x-axis is the same as making a clockwise rotation of θ about the y' axis. This is true because the y'-axis is the negative x-axis ($\hat{\jmath}' = -\hat{\imath}$).

The second equation says that making a counterclockwise rotation of θ about the y-axis is the same as making a counterclockwise rotation of θ about the x' axis. This is true because the x'-axis is the y-axis ($\hat{\imath}' = \hat{\jmath}$).